

**Equations For Bead Pull Cavity Measurements  
LU-159**

**T. G. Jurgens**

**April 4, 1990**

# Equations For Bead Pull Cavity Measurements

## LU-159

T. G. Jurgens

April 4, 1990

### Introduction

This article summarizes the equations pertinent to bead pull cavity measurements for the LINAC upgrade project.

### Symbols

The following symbols are used in this article. The equations are in MKS units.

- $E$  is the electric field at the bead
- $H$  is the magnetic field at the bead
- $E_a$  is the electric field in the unperturbed cavity
- $H_a$  is the magnetic field in the unperturbed cavity
- $Q$  is the quality factor of the unperturbed cavity system
- $U$  is the total stored energy of the unperturbed system
- $W_L$  is the power absorbed by the cavity walls
- $\delta v$  is the volume of the bead
- $\omega_0$  is the resonant frequency of the unperturbed cavity
- $\delta\omega$  is the frequency shift due to the bead
- $\delta\phi$  is the  $S_{12}$  or  $S_{21}$  phase deviation due to the bead

- $\epsilon_0$  is the permittivity of free space
- $\mu_0$  is the permeability of free space
- $\epsilon_r$  is the relative permittivity of the dielectric bead
- $\mu_r$  is the relative permeability of the dielectric bead
- $V$  is the voltage between the end points of a specified integration path
- $R_0$  is the shunt resistance
- $L$  is the structure length

### Equations

One goal, in the design of a LINAC section of coupled cavities, is to have a constant magnitude electric field along the beam axis. The most common way to measure this field is to pull a bead through the beam pipe. A bead passing through a cavity changes its resonant frequency. Beads of various shapes and composition have different effects on this frequency deviation. Equations for spherical beads are given below. Equations for beads of other shapes can be found in the literature [1-4]. For a spherical dielectric bead at least one diameter from a cavity wall the fractional frequency deviation is given by:

$$\frac{\delta\omega}{\omega_0} = -\frac{3\delta v}{4U} \left( \frac{\epsilon_r - 1}{\epsilon_r + 2} \epsilon_0 E^2 + \frac{\mu_r - 1}{\mu_r + 2} \mu_0 H^2 \right) \quad (1)$$

For a spherical perfectly conducting bead at least one diameter from a cavity wall:

$$\frac{\delta\omega}{\omega_0} = -\frac{3\delta v}{4U} \left( \epsilon_0 E^2 + \frac{1}{2} \mu_0 H^2 \right) \quad (2)$$

The above equations include field enhancement effects due to the presence of the bead. The above cited literature also treats the case of beads near a cavity wall. The stored energy of the cavity system, for a given mode, is

$$U = \frac{1}{4} \int_{\text{volume}} (\epsilon E_a^2 + \mu H_a^2) dv \quad (3)$$

The stored energy is related to the power loss in the cavity walls by

$$U = \frac{QW_L}{\omega_0} \quad (4)$$

When excited in the  $\pi/2$  mode, the nose gap of a coupled cavity accelerating structure has high capacitance and is predominantly an electric field region. Therefore, for a perfectly conducting bead equation 2 can be simplified to

$$\frac{\delta\omega}{\omega_0} = -\frac{3\delta v}{4U}\epsilon_0 E^2 \quad (5)$$

The voltage across the nose gap of one cell (cavity) is defined as

$$V = \int_{\text{onecell}} \vec{E} \cdot d\vec{l} \quad (6)$$

The limits of integration of equation 6 are from the beginning of the cavity to the end. For side coupled cavities, these limits are the center lines of the cavity webs. Solving equation 5 for the relative  $E$  field squared results in

$$\frac{E^2}{U} = -\frac{\delta\omega}{\omega_0} \frac{4}{3\epsilon_0\delta v} \quad (7)$$

For a path along the beam axis of a cavity, combining equations 6 and 7 gives the relative voltage across a nose gap.

$$\frac{V}{\sqrt{U}} = \int_{\text{onecell}} \left( -\frac{\delta\omega}{\omega_0} \frac{4}{3\epsilon_0\delta v} \right)^{1/2} dl \quad (8)$$

The relative voltage across the entire structure is obtained by summing the contributions from individual cells.

$$\frac{V}{\sqrt{U}} = \sum_{\text{allcells}} \left[ \int_{\text{onecell}} \left( -\frac{\delta\omega}{\omega_0} \frac{4}{3\epsilon_0\delta v} \right)^{1/2} dl \right] \quad (9)$$

For high  $Q$  systems (  $> 50$  ), the relationship between the frequency shift and the phase deviation is

$$\frac{\delta\omega}{\omega_0} = \frac{1}{2Q} \tan(\delta\phi) \quad (10)$$

To use phase measurement data, equations 8 and 10 can be combined to give the relative voltage across a gap.

$$\frac{V}{\sqrt{U}} = \int_{\text{onecell}} \left( -\frac{1}{2Q} \tan(\delta\phi) \frac{4}{3\epsilon_0\delta v} \right)^{1/2} dl \quad (11)$$

Equation 11 can be summed to get an expression for relative voltage across the entire structure.

$$\frac{V}{\sqrt{U}} = \sum_{\text{all cells}} \left[ \int_{\text{one cell}} \left( -\frac{1}{2Q} \tan(\delta\phi) \frac{4}{3\epsilon_0 \delta v} \right)^{1/2} dl \right] \quad (12)$$

The shunt resistance, in ohms, is

$$R_0 = \frac{V^2 Q}{2\omega_0 U} \quad (13)$$

Substituting the square of equation 9 into equation 13 results in an expression for the shunt resistance of the entire structure in terms of fractional frequency detuning.

$$R_0 = \frac{Q}{2\omega_0} \left[ \sum_{\text{all cell}} \left( \int_{\text{one cell}} \left( -\frac{\delta\omega}{\omega_0} \frac{4}{3\epsilon_0 \delta v} \right)^{1/2} dl \right) \right]^2 \quad (14)$$

Equation 10 and 14 can be combined to give the shunt resistance of the entire structure in terms of phase measurement data.

$$R_0 = \frac{Q}{2\omega_0} \left[ \sum_{\text{all cells}} \left( \int_{\text{one cell}} \left( -\frac{1}{2Q} \tan(\delta\phi) \frac{4}{3\epsilon_0 \delta v} \right)^{1/2} dl \right) \right]^2 \quad (15)$$

In order to calculate shunt resistance in units of ohms per meter, the results of equations 14 or 15 are divided by the structure length,  $L$ .

## References

1. L. C. Maier Jr. and J. C. Slater, Field Strength Measurements in Resonant Cavities, *Jour. Appl. Phys.*, vol. 23, no. 1, (Jan. 1952).
2. S. W. Kitchen and A. D. Schelberg, Resonant Cavity Field Measurements, *Jour. Appl. Phys.*, vol. 26, no. 5, (May 1955).
3. L. B. Mullett, Perturbation of a Resonator, *Atomic Energy Research Establishment*, G/R 853, (Feb. 1952).
4. C. C. Johnson, *Field and Wave Electrodynamics*, McGraw-Hill 1965, pp.229-236.